JOURNAL OF ALGEBRA **205**, 317–325 (1998)
ARTICLE NO. JA977376

On the Dimensions of Cyclic Symmetry Classes of Tensors

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Submitted by Walter Feit

Received November 17, 1997

The dimensions of the symmetry classes of tensors, associated with a certain cyclic subgroup of S_m which is generated by a product of disjoint cycles is explicitly given in terms of the generalized Ramanujan sum. These dimensions can also be expressed as the Euler φ-function and the Möbius function.

1. INTRODUCTION

Let V be an n-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{m}{\otimes} V$ be the mth tensor power of V and write $v_1 \otimes \cdots \otimes v_m$ for the decomposable tensor product of the indicated vectors. To each permutation σ in S_m there corresponds a unique linear operator $P(\sigma)\colon \overset{m}{\otimes} V \to \overset{m}{\otimes} V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$. Let G be a subgroup of S_m and let $I(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\left\{ T(G,\chi)\colon \overset{m}{\bigotimes} V \to \overset{m}{\bigotimes} V \,\middle|\, T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma);\ \chi \in I(G) \right\}$$

is a set of annihiliating idempotents which sum to the identity. The image

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0021-8693/98 $25.00

of $\overset{m}{\otimes} V$ under the $T(G, \chi)$ is called the *symmetry class of tensors* associated with G and χ and is denoted by $V_\chi^m(G)$. It is well known that

$$\dim V_\chi^m(G) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) n^{c(\sigma)}, \tag{1}$$

where $c(\sigma)$ is the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of σ (see [4]).

Several papers are devoted to calculating $\dim V_\chi^m(G)$ in a more closed form than (1). Cummings [2] in the case that G is a cyclic subgroup of $\$_m$ generated by a cycle of length m gives a formula for $\dim V_\chi^m(G)$ in terms of the Euler φ-function and considers the case that G is isomorphic to a direct product of cyclic groups as well. In [3] when G is the dihedral group of order $2m$ is considered and a formula is given when G is equal to the whole group $\$_m$ in [5] and [6]. In all cases $\dim V_\chi^m(G)$ involves certain functions of n.

In [7] there is a formula for calculating $\dim V_\chi^m(G)$ in the case that $G = \langle \pi_1 \rangle \cdots \langle \pi_p \rangle$, where π_is, $1 \leq i \leq p$, are disjoint cycles in $\$_m$. This formula involves the Euler φ-function and Möbius function and is a modification of the formula given in [2]. In this case if the order of π_i is m_i, $1 \leq i \leq p$, then $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_p}$. It is mentioned in [2] and [7] that if two groups are isomorphic, then the dimensions of the symmetry classes of tensors associated with them need not be equal. For example if $G = \langle (12) \rangle$ and $H = \langle (12)(34) \rangle$ are considered as subgroups of $\$_4$, then it easy to calculate that $\dim V_{\chi_0}^4(G) = n^3(n+1)/2$ and $\dim V_{\chi_0}^4(H) = n^2(n^2 + 1)/2$ whereas $G \cong H \cong \mathbb{Z}_2$ and χ_0 is the identity character of $\mathbb{Z}_2$.

Now it is a natural question to consider the cyclic group $G = \langle \pi_1 \cdots \pi_p \rangle$ where the π_is, $1 \leq i \leq p$, are disjoint cycles and ask about the dimension of $V_\chi^m(G)$, where $\chi \in I(G)$. In this case if the order of π_i, $1 \leq i \leq p$, is m_i, then $G \cong \mathbb{Z}_{[m_1, \ldots, m_p]}$, where $[m_1, \ldots, m_p]$ denotes the least common multiple of the integers $m_1, \ldots, m_p$. In this paper we obtain a formula for $\dim V_\chi^m(G)$ in the above case and this formula involves the generalized Ramanujan sum which itself involves the Euler φ-function and Möbius function. There for the rest of this paper let $G < \$_m$ be of the form

$$G = \langle \pi_1 \cdots \pi_p \rangle,$$

where $\pi_i s$, $1 \leq i \leq p$, are disjoint cycles in $\$_m$ of certain orders, say, $m_1, \ldots, m_p$, respectively. Since G is cyclic, therefore the irreducible characters of G are all linear and are of the form

$$\chi_h : G \to \mathbb{C}^*, \qquad \chi_h\left((\pi_1 \ldots \pi_p)^t \right) = \exp\left(\frac{2\pi i h t}{[m_1, \ldots, m_p]} \right),$$
$$0 \leq t \leq [m_1, \ldots, m_p] - 1,$$

so

$$I(G) = \left\{ \chi_h : G \to \mathbb{C}^* \mid 0 \le h \le [m_1, \ldots, m_p] - 1 \right\}, \quad \text{where } [m_1, \ldots, m_p]$$

denotes the least common multiple of the integers $m_1, \ldots, m_p$. The symbol $(m_1, \ldots, m_p)$ denotes the greatest common divisor of $m_1, \ldots, m_p$.

2. A RESULT ABOUT THE RAMANUJAN SUM

The well known Ramanujan sum is

$$C_m(h) = \sum_{\substack{t=0 \\ (t,m)=1}}^{m-1} \exp\left(\frac{2\pi i h t}{m} \right),$$

where m is a positive integer and h is a nonnegative integer. Ramanujan proved that (see [1])

$$C_m(h) = \frac{\varphi(m)\mu(m/(m,h))}{\varphi(m/(m,h))},$$

where φ is the Euler φ-function, i.e., $\varphi(1) = 1$; for $m > 1$, $\varphi(m) =$ the number of positive integers less than m and relatively prime to m, and μ is the Möbius function, i.e., $\mu(1) = 1$, $\mu(m) = 0$ if $p^2 | m$ for some prime number p, and $\mu(m) = (-1)^r$ if $m = p_1 \cdots p_r$, where $p_1, \ldots, p_r$ are distinct prime numbers.

For our main result, we need to generalize the Ramanujan sum. It seems natural to us to generalize the Ramanujan sum as follows.

DEFINITION 1. Let $m_1, \ldots, m_p$ be positive integers and let h be a nonnegative integer. Suppose $d_1 | m_1, \ldots, d_p | m_p$. The *generalized Ramanujan sum* denoted by $S(h; m_1, \ldots, m_p; d_1, \ldots, d_p)$ is defined by

$$S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \sum_{\substack{t=0 \\ (t,m_1)=d_1 \\ \vdots \\ (t,m_p)=d_p}}^{[m_1, \ldots, m_p]-1} \exp\left(\frac{2\pi i h t}{[m_1, \ldots, m_p]} \right)$$

If the set $\{0 \le t \le [m_1, \ldots, m_p] - 1 \mid (t, m_i) = d_i;\ 1 \le i \le p\}$ is empty, then we define $S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = 0$.

Remark 1. It is obvious that $S(h; m; 1) = C_m(h)$, and so the sum appearing in Definition 1 is a generalization of the Ramanujan sum.

In the following lemma we prove that the generalized Ramanujan sum defined in Definition 1 involves the Ramanujan sum.

LEMMA 1. *Let $m_1, \ldots, m_p$ be positive integers and let h be a nonnegative integer. Suppose $d_1 | m_1, \ldots, d_p | m_p$ and set $m'_i = m_i / d_i$, $M_i = m_1 \cdots m_p / m_i$, $M'_i = m'_1 \cdots m'_p / m'_i$, $D_i = d_1 \cdots d_p / d_i$ $(1 \le i \le p)$ and*

$$l = \frac{(M_1, \ldots, M_p)}{(M'_1, \ldots, M'_p)(D_1, \ldots, D_p)}.$$

Then we have

$$S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \begin{cases} \dfrac{1}{l} C_{[m'_1, \ldots, m'_p]}(hl), & \text{if } \left(\dfrac{[d_1, \ldots, d_p]}{d_i}, m'_i \right) = 1, \quad 1 \le i \le p \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Definition 1 and the fact that $\exp(2\pi i h t / [m_1, \ldots, m_p])$ is a periodic function of t with period $[m_1, \ldots, m_p]$ we have

$$S(h; m_1, \ldots, m_p; d_1, \ldots, d_p) = \sum_{\substack{t=0 \\ (t, m_1) = d_1 \\ \vdots \\ (t, m_p) = d_p}}^{[m_1, \ldots, m_p] - 1} \exp\left(\frac{2\pi i h t}{[m_1, \ldots, m_p]} \right)$$

$$= \frac{1}{l} \sum_{\substack{t=0 \\ (t, m_1) = d_1 \\ \vdots \\ (t, m_p) = d_p}}^{l[m_1, \ldots, m_p] - 1} \exp\left(\frac{2\pi i h t}{[m_1, \ldots, m_p]} \right).$$

Now letting $t = [d_1, \dots, d_p]t'$, we obtain

$$S(h; m_1, \dots, m_p; d_1, \dots, d_p) = \frac{1}{l} \sum_{\substack{t'=0 \\ ([d_1,\dots,d_p]t', m_1)=d_1 \\ \vdots \\ ([d_1,\dots,d_p]t', m_p)=d_p}}^{[m'_1,\dots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1, \dots, m'_p]}\right)$$

$$= \frac{1}{l} \sum_{\substack{t'=0 \\ (([d_1,\dots,d_p]/d_1)t', m'_1)=1 \\ \vdots \\ (([d_1,\dots,d_p]/d_p)t', m'_p)=1}}^{[m'_1,\dots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1, \dots, m'_p]}\right).$$

If $([d_1, \dots, d_p]/d_i, m'_i) = 1$ for all i, $1 \le i \le p$, then the set of all the t's indexing the above summation is equal to the set of all the t's such that $0 \le t' \le [m'_1, \dots, m'_p] - 1$ with conditions $(t', m'_i) = 1$, $1 \le i \le p$. And if there is an i for which $([d_1, \dots, d_p]/d_i, m'_i) \ne 1$ then the above sum is zero and therefore we obtain:

$$S(h; m_1, \dots, m_p; d_1, \dots, d_p)$$

$$= \begin{cases} \dfrac{1}{l} \displaystyle\sum_{\substack{t'=0 \\ (t', m'_1)=1 \\ \vdots \\ (t', m'_p)=1}}^{[m'_1,\dots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1, \dots, m'_p]}\right), \\ \qquad \text{if } \left(\dfrac{[d_1, \dots, d_p]}{d_i}, m'_i\right) = 1, \quad 1 \le i \le p \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$= \begin{cases} \dfrac{1}{l} \displaystyle\sum_{\substack{t'=0 \\ (t', [m'_1,\dots,m'_p])=1}}^{[m'_1,\dots,m'_p]-1} \exp\left(\frac{2\pi i h l t'}{[m'_1, \dots, m'_p]}\right), \\ \qquad \text{if } \left(\dfrac{[d_1, \dots, d_p]}{d_i}, m'_i\right) = 1, \quad 1 \le i \le p \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$= \begin{cases} \dfrac{1}{l} C_{[m'_1,\ldots,m'_p]}(hl), & \text{if } \left(\dfrac{[d_1,\ldots,d_p]}{d_i}, m'_i \right) = 1, \quad 1 \le i \le p \\ 0, & \text{otherwise.} \end{cases}$$

∎

In some special cases the generalized Ramanujan sum is given in the following examples:

EXAMPLE 1.

$$S(0;m;d) = C_{m/d}(0) = \frac{\varphi(m/d)\mu((m/d)/(m/d,0))}{\varphi((m/d)/(m/d,0))} = \varphi(m/d).$$

EXAMPLE 2. If $(h,m) = 1$; we have

$$S(h;m;d) = C_{m/d}(h) = \frac{\varphi(m/d)\mu((m/d)/(m/d,h))}{\varphi((m/d)/(m/d,h))} = \mu(m/d).$$

3. THE DIMENSIONS OF SOME SYMMETRY CLASSES OF TENSORS

In this section, as we mentioned earlier, the group $G = \langle \pi_1 \ldots \pi_p \rangle$ is considered, where the π_is, $1 \le i \le p$, are disjoint cycles in $\$_m$. Our aim is to calculate $\dim V_\chi^m(G)$, where $\chi \in I(G)$, in terms of known functions. Our formula involves the generalized Ramanujan sum.

THEOREM 1. *Let $G = \langle \pi_1 \cdots \pi_p \rangle$, where the π_is, $1 \le i \le p$, are disjoint cycles in $\$_m$ of orders $m_1,\ldots,m_p$, respectively, and let χ_h, $0 \le h \le [m_1,\ldots,m_p] - 1$, be an irreducible complex character of G. Then*

$$\dim V_{\chi_h}^m(G) = \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1,\ldots,m_p]} \times \sum_{\substack{d_1|m_1 \\ \vdots \\ d_p|m_p}} S(h;m_1,\ldots,m_p;d_1,\ldots,d_p) n^{d_1+\cdots+d_p},$$

where $S(h; m_1, \dots, m_p; d_1, \dots, d_p)$ *denotes the generalized Ramanujan sum.*

Proof. According to (1) the dimension of $V_{\chi_h}^m(G)$ is

$$\frac{1}{[m_1, \dots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma) n^{c(\sigma)}, \tag{2}$$

where $c(\sigma)$ denotes the number of cycles, including cycles of length 1, in the disjoint cycle decomposition of σ. But every $\sigma \in G$ is equal to $\sigma = (\pi_1 \cdots \pi_p)^t$ for some t, $0 \le t \le [m_1, \dots, m_p] - 1$. since $\pi_1, \dots, \pi_p$ are disjoint, so are $(\pi_1 \cdots \pi_p)^t = \pi_1^t \cdots \pi_p^t$. Appealing to [7] we can obtain

$$c\left(\pi_1^t \cdots \pi_p^t\right) = c\left(\pi_1^t\right) + \cdots + c\left(\pi_p^t\right) + m - (m_1 + \cdots + m_p).$$

Note that if $(t, m_i) = d$, then π_i^t has d cycles of length m_i/d and therefore $c(\pi_i^t) = d = (t, m_i)$. So we have

$$c\left(\pi_1^t \cdots \pi_p^t\right) = (t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p).$$

Hence according to (2) we have

$$\begin{aligned}
&\dim V_{\chi_h}^m(G) \\
&\quad = \frac{1}{[m_1, \dots, m_p]} \sum_{\sigma \in G} \chi_h(\sigma) n^{c(\sigma)} \\
&\quad = \frac{1}{[m_1, \dots, m_p]} \sum_{t=0}^{[m_1, \dots, m_p]-1} \chi_h\left(\pi_1^t \cdots \pi_p^t\right) n^{c(\pi_1^t \cdots \pi_p^t)} \\
&\quad = \frac{1}{[m_1, \dots, m_p]} \\
&\qquad \times \sum_{t=0}^{[m_1, \dots, m_p]-1} \exp\left(\frac{2\pi i h t}{[m_1, \dots, m_p]}\right) n^{(t, m_1) + \cdots + (t, m_p) + m - (m_1 + \cdots + m_p)} \\
&\quad = \frac{n^{m - (m_1 + \cdots + m_p)}}{[m_1, \dots, m_p]} \sum_{t=0}^{[m_1, \dots, m_p]-1} \exp\left(\frac{2\pi i h t}{[m_1, \dots, m_p]}\right) n^{(t, m_1) + \cdots + (t, m_p)}
\end{aligned}$$

Now letting $(t, m_i) = d_i$, $1 \le i \le p$, we obtain

$$\dim V_{\chi_h}^m(G)$$

$$= \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1,\ldots,m_p]} \sum_{\substack{d_1|m_1\\ \vdots \\ d_p|m_p}} \left(\sum_{\substack{t=0\\ (t,m_1)=d_1 \\ \vdots \\ (t,m_p)=d_p}}^{[m_1,\ldots,m_p]-1} \exp\left(\frac{2\pi iht}{[m_1,\ldots,m_p]} \right) \right) n^{d_1+\cdots+d_p}$$

$$= \frac{n^{m-(m_1+\cdots+m_p)}}{[m_1,\ldots,m_p]} \sum_{\substack{d_1|m_1\\ \vdots \\ d_p|m_p}} S(h; m_1,\ldots,m_p; d_1,\ldots,d_p) n^{d_1+\cdots+d_p}.$$

∎

Using Theorem 1 we obtain Theorems 1 and 2 of [2] in the following corollaries.

COROLLARY 1. If G is a cyclic subgroup of $\$_m$ generated by an m-cycle and χ is the identity character 1, then $\dim V_\chi^m(G) = (1/m) \sum_{d\,|\,m} \varphi(m/d)n^d$.

Proof. Since $\chi = \chi_0$, by Example 1 and using Theorem 1 we obtain

$$\dim V_\chi^m(G) = \frac{n^{m-m}}{m} \sum_{d|m} S(0; m; d) n^d = \frac{1}{m} \sum_{d|m} \varphi(m/d) n^d.$$

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Remark 2. In Corollary 1, if $\dim V = n = 1$, then $\dim \overset{m}{\otimes} V = 1$, and so $\dim V_\chi^m(G) = 0$ or 1. so $(1/m)\Sigma_{d\,|\,m}\, \varphi(m/d) = 0$ or 1. But $(1/m)\Sigma_{d\,|\,m}\, \varphi(m/d) = 0$ is impossible, therefore $(1/m)\Sigma_{d\,|\,m}\, \varphi(m/d) = 1$ or $\Sigma_{d\,|\,m}\, \varphi(d) = m$, which is well known identity in number theory.

COROLLARY 2. *If G is a cyclic subgroup of $\$_m$ generated by an m-cycle and χ is a primitive linear character, then* $\dim V_\chi^m(G) = (1/m) \Sigma_{d\,|\,m}\, \mu(m/d)n^d$.

Proof. We know that a linear character of a cyclic subgroup of $\mathfrak{S}_m$ is primitive if its value on a generator of the subgroup is a primitive mth root of unity, so $\chi = \chi_h$ where $(h, m) = 1$ and by Example 2 and using Theorem 1, we have

$$\dim V_\chi^m(G) = \frac{n^{m-m}}{m} \sum_{d|m} S(h; m; d) n^d = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) n^d.$$

∎

EXAMPLE 3. Let $G = \langle(12)(34)(5678)\rangle$ be a subgroup of $\mathfrak{S}_9$. Suppose χ is the identity character 1, i.e., $\chi = \chi_0$. Then by Theorem 1 we have

$$\begin{aligned}\dim V_\chi^9(G) &= \frac{n^{9-(2+2+4)}}{4} \sum_{\substack{d_1|2\\ d_2|2\\ d_3|4}} S(0; 2, 2, 4; d_1, d_2, d_3) n^{d_1+d_2+d_3}\\ &= \frac{n}{4}\big[S(0; 2, 2, 4; 2, 2, 4) n^8 + S(0; 2, 2, 4; 2, 2, 2) n^6\\ &\qquad + S(0; 2, 2, 4; 1, 1, 1) n^3\big]\\ &= \frac{n}{4}[n^8 + n^6 + 2n^3].\end{aligned}$$

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